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1/2 BPS correlator and free fermion

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Abstract: We propose that in the BMN limit the effective interaction vertex in the 1/2 BPS sector of $\mathcal{N} = 4$ SYM is given by the Das-Jevicki-Sakita hamiltonian. We check for some examples that it reproduces the $1/N$ correction to the correlation functions of 1/2 BPS operators.

Keywords: Matrix Models, Penrose limit and pp-wave background, 1/N Expansion, Supersymmetric gauge theory.

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Contents

1. Introduction

In [1, 2], it was suggested that the 1/2 BPS sector of $\mathcal{N}=4$ SYM is described as a matrix quantum mechanics in a harmonic potential. It is well-known that the Hilbert space of this system is represented by a free fermion system. In a recent paper [3], the gravity dual of these 1/2 BPS states are constructed and they are indeed characterized by the incompressible Fermi fluid in a two-dimensional plane.

The 1/2 BPS operators are given by the product of traces of the scalar field $Z = X_1 + iX_2$ in $\mathcal{N}=4$ SYM. Their correlation functions do not receive quantum corrections, so they are given by

$$\left\langle \prod_{i=1}^{k} \mathrm{Tr}\, Z^{J_i}(x) \prod_{j=1}^{l} \mathrm{Tr}\, \overline{Z}^{K_j}(y) \right\rangle_{\mathcal{N}=4\ \mathrm{SYM}} = \frac{G_{\{J_i\}\{K_j\}}}{|x-y|^{2J}} \tag{1.1}$$

where J is the total U(1) charge

$$J = \sum_{i=1}^{k} J_i = \sum_{j=1}^{l} K_j, \tag{1.2}$$

and it is equal to the conformal dimension of the 1/2 BPS operator. The non-trivial information is solely contained in the numerical factor $G_{\{J_i\}\{K_j\}}$ in (1.1). It was shown in [4, 5] that this factor is written as a complex gaussian matrix integral

$$G_{\{J_i\}\{K_j\}} = \int [dZd\overline{Z}] e^{-\mathrm{Tr}(Z\overline{Z})} \prod_{i=1}^{k} \mathrm{Tr}\, Z^{J_i} \prod_{j=1}^{l} \mathrm{Tr}\, \overline{Z}^{K_j}. \tag{1.3}$$

It is further argued that in the BMN limit [6]

$$N, J \to \infty \quad \text{with} \quad g_2 = \frac{J^2}{N} \quad \text{fixed} \tag{1.4}$$

the non-planar diagrams survive in the computation of $G_{\{J_i\}\{K_j\}}$ and it becomes a non-trivial function of g_2.

In this short note, we try to connect the above two facts. We argue that the BMN limit of $G_{\{J_i\}\{K_j\}}$ can be computed from the free fermion picture. This paper is organized as follows. In section 2, we review the Schur polynomial as the orthogonal basis of 1/2 BPS operators and their relation to the free fermions. In section 3, we propose that the BMN limit of two-point function can be reproduced from the Das-Jevicki-Sakita hamiltonian. section 4 is discussions.

2. Schur polynomial and free fermion

In this section, we review the relation between 1/2 BPS operators and the Schur polynomials [1, 2]. To write down the 1/2 BPS operators, it is useful to introduce the free boson α_n obeying the standard commutation relation

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}. \tag{2.1}$$

Then we introduce the coherent state

$$|Z\rangle = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr} Z^n \alpha_{-n}\right)|0\rangle \tag{2.2}$$

which satisfies

$$\alpha_J|Z\rangle = \operatorname{Tr} Z^J|Z\rangle. \tag{2.3}$$

In other words, the oscillator α_J corresponds to a single trace operator $\operatorname{Tr} Z^J$. In particular the mode number of oscillator corresponds to the length of the trace, which in turn is identified as the length of string via the spin chain picture. More generally, the multi-trace operators correspond to the product of boson oscillators

$$\prod_{i=1}^{k} \alpha_{J_i}|Z\rangle = \prod_{i=1}^{k} \operatorname{Tr} Z^{J_i}|Z\rangle. \tag{2.4}$$

The basis of operators (2.4) is not diagonal with respect to the two-point function. The diagonal basis is obtained by fermionizing the boson α_n

$$\alpha_n = \sum_{r\in\mathbb{Z}+\frac{1}{2}} c_{n-r}b_r \tag{2.5}$$

where b_r and c_r obey the anti-commutation relation

$$\{c_r, b_s\} = \delta_{r+s,0}. \tag{2.6}$$

For a given Young diagram R, we introduce the state $\langle R|$ as

$$\langle R| = \langle 0| \prod_{i=1}^{s} c_{h_i-i+1/2} \prod_{j=1}^{s} b_{v_j-j+1/2} \tag{2.7}$$

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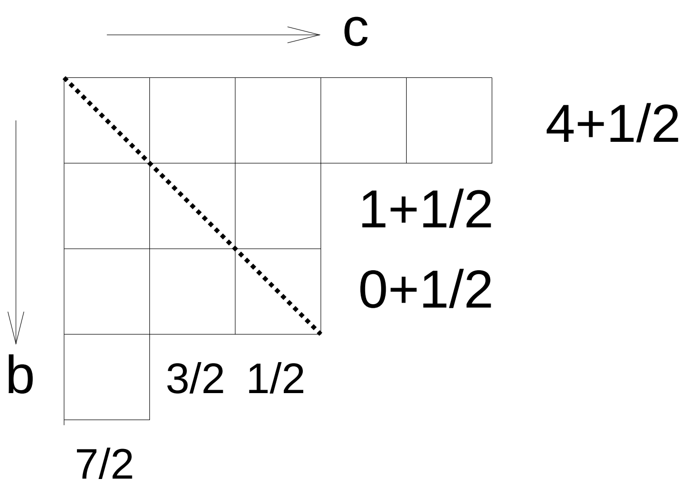

Figure 1: Young Diagram and Fermions. The diagram is split into two parts by the diagonal line (dashed line in the figure). The mode number r of c_r and b_r corresponds to the (number of boxes)+1/2 in the direction shown by the arrows. The number of boxes along the diagonal is s. ($s = 3$ in this figure.)

where h_i are the row-lengths and v_j are the column-lengths, and s is the number of boxes along the diagonal. For example, the diagram in the figure 1 corresponds to the state

$$\Big\langle \yng(5,3,3,1) \Big| = \langle 0|c_{9/2}c_{3/2}c_{1/2}b_{7/2}b_{3/2}b_{1/2}. \tag{2.8}$$

In terms of the state $\langle R|$ in (2.7), the Schur polynomial is defined by

$$\langle R|Z\rangle = S_R(Z). \tag{2.9}$$

This basis is diagonal with respect to the gaussian measure for the $N \times N$ complex matrix Z

$$\int [dZd\overline{Z}]e^{-\operatorname{Tr}(Z\overline{Z})} S_R(Z) S_{R'}(\overline{Z}) = \mathcal{N}_R \delta_{R,R'}. \tag{2.10}$$

For instance, at the level $L_0 = 2$ there are two states

$$\langle \yng(2)| = \langle 0|c_{3/2}b_{1/2}, \qquad \langle \yng(1,1)| = \langle 0|c_{1/2}b_{3/2} \tag{2.11}$$

and the corresponding Schur polynomial is given by

$$\langle \yng(2)|Z\rangle = \frac{1}{2}\Big[\operatorname{Tr} Z^2 + (\operatorname{Tr} Z)^2\Big], \qquad \langle \yng(1,1)|Z\rangle = \frac{1}{2}\Big[\operatorname{Tr} Z^2 - (\operatorname{Tr} Z)^2\Big]. \tag{2.12}$$

In the computation of overlap integral (2.10), the following kernel naturally appears

$$e^{\frac{1}{2N}\mathcal{V}_{\rm int}} \equiv \int [dZd\overline{Z}]e^{-\operatorname{Tr}(Z\overline{Z})} |Z\rangle\langle Z|. \tag{2.13}$$

Once we know the "interaction vertex" $\mathcal{V}_{\rm int}$ defined by (2.13), it is straightforward to compute the overlap integral in the free boson picture

$$\langle 0| \prod_{i=1}^{k} \alpha_{J_i} e^{\frac{1}{2N}\mathcal{V}_{\rm int}} \prod_{j=1}^{l} \alpha_{-K_j}|0\rangle = \int [dZd\overline{Z}]e^{-\operatorname{Tr}(Z\overline{Z})} \prod_{i=1}^{k} \operatorname{Tr} Z^{J_i} \prod_{j=1}^{l} \operatorname{Tr} \overline{Z}^{K_j}. \tag{2.14}$$

3. $\mathcal{V}_{\rm int}$ as Das-Jevicki-Sakita hamiltonian

Although it is known how to perform the gaussian matrix integral [7], it is not so easy to evaluate the Z-integral (2.13) in a simple form and write down the interaction vertex $\mathcal{V}_{\rm int}$ in terms of oscillators α_n. However, we expect that $\mathcal{V}_{\rm int}$ simplifies in some particular limit. We propose that in the BMN limit $\mathcal{V}_{\rm int}$ can be replaced by the Das-Jevicki-Sakita hamiltonian $\mathcal{V}_3$

$$\mathcal{V}_3 = \sum_{n,m>0} \alpha_{-n}\alpha_{-m}\alpha_{n+m} + \alpha_{-n-m}\alpha_n\alpha_m. \tag{3.1}$$

This is motivated by the intuition that string interaction can be written as the splitting/joining and the length of strings is conserved in the BMN limit (see figure 2). This vertex can be thought of as the lightcone string field interaction in the 1/2 BPS sector of $\mathcal{N}=4$ SYM.

The important property of DJS hamiltonian $\mathcal{V}_3$ is that it is diagonal in the fermion basis (i.e. $\mathcal{V}_3$ is a W-current)

$$\mathcal{V}_3 = \sum_r r^2 c_{-r} b_r. \tag{3.2}$$

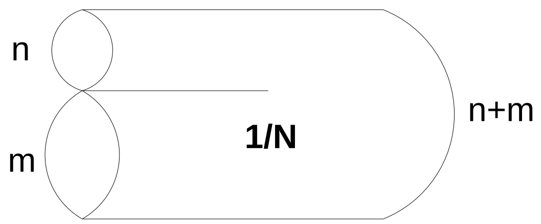

Figure 2: This diagram represents the splitting and joining of strings. In this process the $\mathrm{U}(1)_J$ charge, or the total length of strings is conserved. This diagram comes with a factor of $N^\chi = N^{-1}$.

We can easily see that $\mathcal{V}_3$ measures the square of the mode number of fermions

$$[\mathcal{V}_3, c_r] = r^2 c_r, \qquad [\mathcal{V}_3, b_r] = -r^2 b_r. \tag{3.3}$$

In particular, $\mathcal{V}_3$ is diagonal in the representation basis and its eigenvalue is given by the second Casimir

$$\langle R| e^{\frac{1}{2N}\mathcal{V}_3} |R'\rangle = e^{\frac{1}{2N}C_2(R)}\delta_{R,R'}. \tag{3.4}$$

To compute the two-point function of states given by the boson basis, we have to rewrite them in terms of the fermions:

$$\begin{aligned}\prod_{i=1}^k \alpha_{-J_i}|0\rangle = &\sum_{S\subset\{1,\cdots,k\}} (-1)^{|S|+1} \sum_{0<r<J_S} c_{-J+r} b_{-r}|0\rangle + \\ &+\frac{1}{2}\sum_{A\cup B=\{1,\cdots,k\}}\sum_{S\subset A}\sum_{T\subset B}(-1)^{|S|+|T|}\sum_{0<r<J_S}\sum_{0<s<J_T} c_{-J_A+r}b_{-r}c_{-J_B+s}b_{-s}|0\rangle + \\ &+\cdots\end{aligned} \tag{3.5}$$

where $|S|$ denotes the number of elements in S, and $J_S = \sum_{i\in S} J_i$. The first line in (3.5) is the state with one particle-hole pair, and the second line is the state with two particle-hole pairs. The dots denote the higher particle-hole pair states.

Below, we check our proposal (3.1) for two examples.

3.1 Example 1: the $1 \to k$ amplitude

Let us consider the process that a single trace operator splits into an operator with k traces, i.e. the amplitude $G_{\{J\}\{J_i\}}$. We would like to show that this amplitude is written by using the DJS hamiltonian $\mathcal{V}_3$ in the free fermion picture

$$\langle 0|\alpha_J e^{\frac{1}{2N}\mathcal{V}_3}\prod_{i=1}^{k}\alpha_{-J_i}|0\rangle = \sum_{S\subset\{1,\cdots,k\}}(-1)^{|S|+1}\sum_{0<r<J_S}\exp\left[\frac{1}{2N}\Big((J-r)^2-r^2\Big)\right]. \tag{3.6}$$

Note that in this process only a one particle-hole pair contributes. Therefore, we used the first line in (3.5) to rewrite the amplitude in terms of the free fermions. On the right hand side of (3.6), we used the relation (3.3). In the BMN limit, the summation over r is replaced by the integral over $x = r/J$

$$\begin{aligned}\langle 0|\alpha_J e^{\frac{1}{2N}\mathcal{V}_3}\prod_{i=1}^{k}\alpha_{-J_i}|0\rangle &\simeq \sum_{S\subset\{1,\cdots,k\}}(-1)^{|S|+1}J\int_0^{J_S/J}dx\exp\left[\frac{J^2}{2N}(1-2x)\right]\\ &= \frac{N}{J}\sum_{S\subset\{1,\cdots,k\}}(-1)^{|S|}\exp\left[\frac{J}{2N}(J_{\overline{S}}-J_S)\right]\\ &= \frac{N}{J}\prod_{i=1}^{k}\left[\exp\left(\frac{JJ_i}{2N}\right)-\exp\left(-\frac{JJ_i}{2N}\right)\right].\end{aligned} \tag{3.7}$$

Here $\overline{S} = \{1,\cdots,k\} - S$ is the complement of S. In the second equality, we used the relation $\sum_S(-1)^{|S|} = 0$. Finally, the BMN limit of this amplitude is given by

$$\lim_{N,J\to\infty,\frac{J^2}{N}=g_2}\langle 0|\alpha_J e^{\frac{1}{2N}\mathcal{V}_3}\prod_{i=1}^{k}\alpha_{-J_i}|0\rangle = \frac{J}{g_2}\prod_{i=1}^{k}2\sinh\left(\frac{g_2}{2}\widehat{J_i}\right) \tag{3.8}$$

where $\widehat{J_i} = J_i/J$.

On the other hand, the corresponding matrix integral is known at finite N [4, 5, 8–10]–,BergereAZ

$$\int[dZd\overline{Z}]e^{-\operatorname{Tr}(Z\overline{Z})}\operatorname{Tr}\overline{Z}^J\prod_{i=1}^{k}\operatorname{Tr}Z^{J_i} = \frac{1}{J+1}\sum_{S\subset\{1,\cdots,k\}}(-1)^{|S|}\frac{\Gamma(N+J_{\bar{S}}+1)}{\Gamma(N-J_S)} \tag{3.9}$$

In the BMN limit (3.9) becomes

$$\lim_{N,J\to\infty,\frac{J^2}{N}=g_2}\int[dZd\overline{Z}]e^{-\operatorname{Tr}(Z\overline{Z})}\operatorname{Tr}\overline{Z}^J\prod_{i=1}^{k}\operatorname{Tr}Z^{J_i} = \frac{JN^J}{g_2}\prod_{i=1}^{k}2\sinh\left(\frac{g_2}{2}\widehat{J_i}\right). \tag{3.10}$$

From (3.8) and (3.10), one can see that the DJS hamiltonian correctly reproduces the matrix integral up to an overall factor N^J. The factor N^J can be taken care of by modifying the identification as

$$e^{\frac{1}{2N}\mathcal{V}_{\rm int}} \simeq N^{L_0}e^{\frac{1}{2N}\mathcal{V}_3}. \tag{3.11}$$

3.2 Example 2: the $2 \to 2$ amplitude

Next example is the $2 \to 2$ amplitude $G_{\{J_1,J_2\}\{K_1,K_2\}}$. The free fermion computation is straightforward as in example 1. In this case, both one particle-hole pair and two particle-hole pairs contribute to the amplitude. So we need the first and the second line (3.5) to rewrite the bosons into fermions. Explicitly, the two-boson state is written in terms of fermions as

$$\alpha_{-K_1}\alpha_{-K_2}|0\rangle = \left(\sum_{0<r<K_1} + \sum_{0<r<K_2} - \sum_{0<r<J}\right) c_{-J+r}b_{-r}|0\rangle + \\ + \sum_{0<r<K_1}\sum_{0<s<K_2} c_{-K_1+r}b_{-r}c_{-K_2+s}b_{-s}|0\rangle. \tag{3.12}$$

After a similar calculation as in example 1, the BMN limit of the $2 \to 2$ amplitude is found to be

$$\lim_{N,J\to\infty,\frac{J^2}{N}=g_2} \langle 0|\alpha_{J_1}\alpha_{J_2}e^{\frac{1}{2N}\mathcal{V}_3}\alpha_{-K_1}\alpha_{-K_2}|0\rangle = \\ = \frac{J}{g_2}2^3 \sinh\left(\frac{g_2}{2}\widehat{J}_1\right)\sinh\left(\frac{g_2}{2}\widehat{J}_2\widehat{K}_1\right)\sinh\left(\frac{g_2}{2}\widehat{J}_2\widehat{K}_2\right). \tag{3.13}$$

Here we assumed that $J_2 = \max\{J_i, K_j\}, J_1 = \min\{J_i, K_j\}$.

On the other hand, the BMN limit of the matrix integral is [10]

$$\lim_{N,J\to\infty,\frac{J^2}{N}=g_2} \int [dZd\overline{Z}]e^{-\operatorname{Tr}(Z\overline{Z})} \operatorname{Tr}\overline{Z}^{J_1}\operatorname{Tr}\overline{Z}^{J_2}\operatorname{Tr}Z^{K_1}\operatorname{Tr}Z^{K_2} = \\ = \frac{JN^J}{g_2}2^3 \sinh\left(\frac{g_2}{2}\widehat{J}_1\right)\sinh\left(\frac{g_2}{2}\widehat{J}_2\widehat{K}_1\right)\sinh\left(\frac{g_2}{2}\widehat{J}_2\widehat{K}_2\right). \tag{3.14}$$

Again, the two computations (3.13) and (3.14) agree up to a factor N^J.

4. Discussion

We have checked for two examples that $\mathcal{V}_{\text{int}}$ defined in (2.13) can be replaced by the DJS hamiltonian $\mathcal{V}_3$ (3.1) in the BMN limit. This agreement strongly suggests that the identification (3.11) holds for the general correlator $G_{\{J_i\}\{K_j\}}$ (1.3). It would be nice to find a general proof. In [12], a similar interaction vertex Σ was introduced in the string bit picture, and it was shown that it reproduces the correct g_2 dependence. It would be interesting to relate their vertex Σ and ours $\mathcal{V}_3$. It is well-known that the DJS hamiltonian naturally appears in the two-dimensional Yang-Mills theory [13, 14]. It would be interesting to find its relation to the BMN limit of 1/2 BPS sector (see [8] for a discussion on the relation of 1/2 BPS correlators and 2d Yang-Mills). Finally, it would be extremely interesting to find a useful description of the 1/4 and 1/8 BPS states (see [15, 16] for some attempts).

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